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Similarity stagnation point solutions of the Navier–Stokes equations – review and extension

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Abstract

Stagnation regions exist on all blunt bodies moving in a viscous fluid. In certain stagnation flow problems, the Navier–Stokes equations reduce to nonlinear ordinary differential equations through a similarity transform. This paper reviews the existing steady similarity stagnation flow solutions and discusses a new area of research, stagnation flow with slip.

Keywords: Navier–Stokes; Stagnation flow; Similarity solutions; Slip

1. Introduction

The Navier–Stokes equations are the basic governing equations of fluid mechanics. This set of nonlinear partial differential equations has no general solution, and analytic solutions are rare. However, in certain flow problems similarity transforms may be possible, reducing the Navier–Stokes equations to a set of nonlinear ordinary differential equations which are much easier to solve. Similarity solutions not only describe fundamental physically significant problems but also serve as accuracy standards for full numerical solutions. Since a previous review 16 years ago [1], many new solutions have appeared, and a new review is necessary.

Similarity transforms, which reduces the number of independent variables for partial differential equations, is possible only for problems with certain physical symmetries. The general theory falls under infinitesimal Lie transformation groups [2,3] which include translations, rotations and stretching deformations. Since only the stretching transform yields all significant solutions, one can use a simpler stretching method described by Hansen [4] and Ames [5].

The present paper is concerned with the similarity solutions of the stagnation flows. This class of solutions, describing the fluid flow near the stagnation region, exists on all solid bodies moving in a fluid. The stagnation region encounters the highest pressure, the highest heat transfer, and the highest rates of mass deposition. We shall first give a short introduction on the basic Hiemenz solution describing stagnation flow towards a solid plate. Then we discuss the various existing extensions of Hiemenz flow. Emphasis is placed on the physical problem and its similarity formulation. Due to the limited objectives of this review, one is directed to the original sources for the ramifications or outcomes of these extensions. Lastly, research into stagnation flows with partial boundary slip is discussed.

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2. The basic Hiemenz solution

The basic stagnation flow solution is due to Hiemenz [6]. Consider the two-dimensional inviscid (irrotational) flow towards a plate at $z = 0$

$$u = ax, \qquad w = -az \tag{1}$$

where (u, υ, w) are velocity components in the Cartesian directions (x, y, z) and a is a proportionality constant. Eq. (1) represents the leading terms of any 2D stagnation flow in the stagnation region. For viscous flow Hiemenz used the similarity transform

$$u = axf'(\xi), \quad w = -\sqrt{a\nu}\, f(\xi), \quad \xi = \sqrt{a/\nu}\, z \tag{2}$$

where ν is the kinematic viscosity of the fluid. Eq. (2) satisfies the continuity equation and the x-momentum Navier–Stokes equation reduces to the nonlinear ordinary differential equation

$$f''' + ff'' - (f')^2 + 1 = 0. \tag{3}$$

The no-slip boundary conditions on the plate are

$$f(0) = 0, \qquad f'(0) = 0. \tag{4}$$

At infinity, the flow approaches the inviscid flow equation (1). Thus

$$f'(\infty) = 1. \tag{5}$$

Accurate numerical integration using a shooting algorithm yields the initial value $f''(0) = 1.232588$. The existence and uniqueness of the solution to Hiemenz stagnation flow was shown by Tam [7], Craven and Peletier [8]. It was found that f' increases monotonically from zero and approaches unity exponentially (e.g. [9]).

Although inviscid flow can be reversed ($a < 0$ in Eq. (1)), similarity solutions do not exist for the corresponding viscous flow where vorticity is transported into the stagnation region and diffuse all over.

The two-dimensional Hiemenz stagnation flow can be extended to axisymmetric and three-dimensional as follows. These extensions are reviewed in [1] but included here for completeness. The 3D inviscid stagnation flow normal to a solid plate at $z = 0$ is given by

$$u = ax, \quad \upsilon = \lambda ay, \quad w = -a(1 + \lambda)z. \tag{6}$$

The similarity transform for viscous flow, satisfying continuity, is

$$u = axf'(\xi), \quad \upsilon = \lambda ayg'(\xi), \quad w = -\sqrt{a\nu}\,(f + \lambda g). \tag{7}$$

Then the Navier–Stokes equations reduce to the nonlinear ordinary differential equations

$$f''' + (f + \lambda g)f'' - (f')^2 + 1 = 0, \tag{8}$$

$$g''' + (f + \lambda g)g'' - \lambda\left[(g')^2 - 1\right] = 0. \tag{9}$$

The boundary conditions are that the inviscid solution is approached at infinity

$$f'(\infty) = 1, \qquad g'(\infty) = 1. \tag{10}$$

If the plate is still, the no slip conditions (without loss of generality) are

$$f(0) = 0, \quad f'(0) = 0, \quad g(0) = 0, \quad g'(0) = 0. \tag{11}$$

The two-dimensional case $\lambda = g = 0$, was solved by Hiemenz. The axisymmetric stagnation flow towards a plate, $\lambda = 1$, $g = f$, was solved by Homann [10]. Howarth [11] studied the case $0 < \lambda < 1$ which can be applied to the stagnation region of an ellipsoid. Davey [12] investigated the stagnation region near a saddle point ($-1 < \lambda < 0$). For $\lambda \leqslant -1$ the vorticity generated is not confined in the boundary layer and the existence of solutions cannot be shown.

The 2D inviscid flow at infinity may be rotational

$$u = ax + bz, \qquad w = -az \tag{12}$$

where a, b are constants. The corresponding similarity transform for viscous flow is

$$u = axf'(\xi) + b\sqrt{\nu/a}\,h(\xi), \qquad w = -\sqrt{a\nu}\,f. \tag{13}$$

Then Eqs. (3)–(5) are the same and in addition

$$h'' + fh' - hf' = 0, \tag{14}$$

$$h(0) = 0, \qquad h'(\infty) = 1. \tag{15}$$

This oblique stagnation flow was first solved by Stuart [13], and again by Tamada [14], Dorrepaal [15]. The general rotational flow near a stagnation point was discussed by Davey [16]. Notice the one-way coupling, that the function f influences the function h but not vice versa.

Another one-way coupling problem is when the plate translates in its own plane. Rott [17] considered the 2D normal stagnation flow towards a translating plate. The extra governing equations are similar to Eqs. (14), (15) except the boundary conditions are

$$h(0) = 1, \qquad h(\infty) = 0. \tag{16}$$

Rott found the rare exact relation

$$h = f''(\xi)/f''(0). \tag{17}$$

The axisymmetric Homann stagnation flow toward a translating plate was done by Wang [18] and also by Libby [19].

3. Stagnation flow with porous boundaries

Stagnation flow towards a porous plate which allows suction or injection models condensation or evaporation from the plate. The conditions in Eq. (4) are relaxed to

$$f(0) = s = -W/\sqrt{a\nu}, \qquad f'(0) = 0 \tag{18}$$

where W is the injection velocity. Such flows were discussed by Jones and Watson [9], Alexander [20], Labropulu et al. [21], Weidman and Mahalingam [22]. Similar to porous channel flows [23,24], multiple solutions for large suction are found to exist [25].

Of different nature is the stagnation flow from a porous disk towards another parallel disk. These flows are important in porous thrust bearings and air cushioned sliders. We limit our review to cases where the flow is primarily stagnation type, although rotation can be included (for primarily rotating flow, see Zandbergen and Dijkstra [26]). Since there is a length scale (the distance between plates d) and a velocity scale (injection velocity W), the similarity transform is

$$u = (W/d)rf'(\eta), \quad w = -2Wf(\eta), \quad \eta = z/d. \tag{19}$$

Here (u, w) are velocity components in the cylindrical (r, z) coordinates. The Navier–Stokes equations become

$$f'''' + 2Rff''' = 0 \tag{20}$$

where $R = Wd/\nu$ is the cross flow Reynolds number. Eq. (20) was studied by Rasmussen [27], Terrill and Cornish [28], Chapman and Bauer [29] numerically and analytically for both large and small R. The existence and uniqueness was proved by Elcrat [30].

The use of porous stagnation flow for air-cushioned lifting and sliding was first considered by Wang [31]. Although the slider is circular, the governing equations were written in Cartesian coordinates to accommodate lateral sliding. Skalak and Wang [32] studied the strip slider, and Watson et al. [33] the elliptic slider. The problem was investigated more recently [34,35], perhaps with no knowledge of previous research.

4. Stagnation flow on a circular cylinder

Axisymmetric stagnation flow on a circular cylinder was first introduced by Wang [36]. The flow far from the cylinder is the axisymmetric inviscid flow

$$u = -a(r - b^2/r), \qquad w = 2az \tag{21}$$

where a is the strength of the stagnation flow and b is the radius of the solid cylinder. For viscous flow set

$$u = -abf(\eta)/\sqrt{\eta}, \quad w = 2azf'(\eta), \quad \eta \equiv (r/b)^2. \tag{22}$$

The Navier–Stokes equations then reduce to

$$\eta f''' + f'' + R\big[1 + ff'' - (f')^2\big] = 0 \tag{23}$$

where $R = ab^2/(2\nu)$ is a Reynolds number. The boundary conditions are

$$f(1) = 0, \quad f'(1) = 0, \quad f'(\infty) = 1. \tag{24}$$

For each value of R, a similarity solution can be integrated form Eqs. (23), (24). As $R \to \infty$, one can recover the Hiemenz equations by the transform

$$f = \sqrt{R}\,\bar{f}(\xi), \qquad \xi = \sqrt{R}\,(\eta - 1). \tag{25}$$

Gorla [37] added a longitudinal translation of the cylinder, which introduces a decoupled equation similar to the translating plate. Cunning et al. [38] considered rotation of the cylinder, in addition to suction/injection on the wall. The existence and uniqueness was studied by Paullet and Weidman [39].

The oblique axisymmetric stagnation flow on a cylinder was solved by Okamoto [40] and again by Weidman and Putkaradze [41]. The flow at infinity is inviscid but rotational, where the longitudinal velocity in Eq. (21) is supplanted by

$$w = 2az + c(r^2 - b^2). \tag{26}$$

5. Stagnation flow on a fluid surface

The stagnation flow of a lighter fluid against a heavier, originally quiescent fluid was investigated by Wang [42]. While the lighter fluid is driven by a far field pressure gradient as in Hiemenz flow, the heavier fluid is dragged along by the interface viscosity. The interface is assumed to be horizontal, maintained by gravity forces. The velocities and shear stresses of the two fluids (both proportional to the distance from the stagnation point) are matched on the interface. Thus the boundary conditions equation (4) are generalized to

$$f(0) = 0, \qquad f'(0) = \beta \tag{27}$$

where β depends on the properties of the two fluids, such as viscosity and density ratios. The corresponding oblique stagnation flow on a quiescent fluid was solved by Liu [43]. On the other hand, if there is only one fluid and β is given, the problem is equivalent to a stagnation flow on a stretching boundary [44].

The impingement of two different stagnation flows was investigated by Wang [45]. Again the properties of the two fluids are important parameters. Davis and Yadigaroglu [46] considered added mass transfer at the interface. The oblique impingement of two stagnations flows was studied by Tilley and Weidman [47].

6. Stagnation flows with partial slip

Recently, stagnation flows with partial slip on the boundary are being studied. Partial slip occurs, for example, on fluoroplastic coating (e.g. Teflon) which resists adhesion. Some surfaces are rough or porous such that equivalent slip occurs [48]. Also, there exists a hydrodynamic boundary slip regime for rarefied gases when the Knudsen number is small [49,50]. In these cases, the no-slip condition is replaced by Navier's condition [51], where the tangential slip velocity u is proportional to the shear stress τ

$$u = N\tau. \tag{28}$$

The stagnation flow on a plate (both normal and oblique) with partial slip was solved recently by Wang [52]. It was found that slip greatly affects the flow field. The existence and asymptotic behavior was discussed by Ishimura and Ushijima [53].

The stagnation slip flow on a on an axially moving cylinder is important, for example, in the cooling of moving rods with a partial slip surface. Consider a cylinder of radius a moving longitudinally with velocity W. The flow far from the cylinder is the axisymmetric potential flow equation (21), but Eq. (22) is replaced by

$$u = -kaf(\eta)/\sqrt{\eta}, \quad w = 2kzf'(\eta) + Wg(\eta), \quad \eta \equiv (r/a)^2. \tag{29}$$

The Navier–Stokes equations then reduce to two nonlinear ordinary differential equations, Eq. (23) and

$$\eta g'' + g' + R(fg' - gf') = 0. \tag{30}$$

The boundary conditions are influenced by slip. Eq. (28) gives

$$w - W = N\rho\nu\frac{\partial w}{\partial r}. \tag{31}$$

Substituting w, Eq. (31) is separated to

$$f'(1) = \lambda f''(1), \qquad g(1) = 1 + \lambda g'(1) \tag{32}$$

where $\lambda \equiv 2N\rho\nu/a$ is a normalized slip factor. Zero penetration on the cylinder gives

$$f(1) = 0. \tag{33}$$

At infinity, the flow tends to the potential flow

$$f'(\infty) = 1, \qquad g(\infty) = 0. \tag{34}$$

We shall not go into the details of the solution here. Note that most of the previous stagnation problems can be extended to include partial slip boundary conditions.

7. Conclusions

Due to the nonlinearity of the Navier–Stokes equations, similarity solutions are rare. The existing relevant solutions, however, are scattered over various publications. This review would serve as a major reference for researchers in this area, such that duplication of efforts would be minimized.

We considered the steady, constant property Navier–Stokes equations. The less restrictive class of boundary-layer solutions are excluded, although some of these solutions do satisfy the Navier–Stokes equations.

What is not covered here include unsteady flows, non-Newtonian fluids, flows in porous media, variable properties, electro-magnetic fluids, natural convection, etc. Certainly the solutions presented herein may be extended to these important areas.

Although the no-slip boundary condition has dominated research for the past centuries, renewed interest on partial slip was generated by advances in materials science and micro-rarefied flow. Some of the recent advances in stagnation flows with partial slip are discussed. Its is hoped that this short review may elicit further research in this important area.

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